most 12% and generally much less than this. The second FET, of course, makes up the balance.

Finally, an alternative but entirely equivalent way of interpreting the above analysis is to consider the load line seen by each FET [5]. Each FET has the same current swing but very different magnitudes and phases of voltage swing which are frequency dependent with the result that each FET operates along a different load line. As observed by Salib *et al.* [5] the load line the FET operates along is the electronic load line not the circuit element value of $Z_\pi/2$.

III. Conclusion

Closed-form analytic expressions for the voltage and power distribution along the drain line of an ideal distributed amplifier have been presented. It has been shown that the power generation is very nonuniform and that some FET's may actually absorb power rather than generate it over a portion of the frequency band.

References

[1] K. B. Niclas, R. R. Pereira, and A. P. Chang, "On power distribution in additive amplifiers," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 1692–1700, Nov. 1990

[2] G. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures*. New York: McGraw-Hill, 1964.

[3] C. S. Aitchison, "The intrinsic noise figure of the MESFET distributed amplifier," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 460–466, June 1985.

[4] Y. Ayasli, S. W. Miller, R. Mozzi, and L. K. Hanes, "Capacitively coupled travelling-wave power amplifier," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 1704–1709, Dec. 1984.

[5] M. L. Salib, D. E. Dawson, and H. K. Hahn, "Load-line analysis in the frequency domain with distributed amplifier design examples," in *IEEE MTT-S Int. Microwave Symp. Dig.*, June 1987, pp. 575–578.

A Convergence Acceleration Procedure for Computing Slowly Converging Series

Surendra Singh and Ritu Singh

***Abstract*—The application of sloped θ-algorithm to the partial sums of a slowly converging series is shown to accelerate its convergence. The algorithm is applied to accelerate the convergence of series representing the free-space periodic Green's functions involving the zeroth-order Hankel function of the second kind, and its associated Fourier transform. Numerical results indicate that the algorithm converges faster than the Shanks' transform. It is also able to sum the series to machine precision in about 20 terms. A relative error measure is shown as a function of the number of terms of various combinations of source and observation points. The relative saving in computation time is also provided to show the benefit of using the algorithm.**

I. Introduction

In the numerical solution of problems involving a periodic geometry, one is usually faced with repeated evaluations of an infinite series represented the Green's function. The Green's function series converges very slowly. Hence, a significant amount of computer CPU time is spent in repeated computations of such series. One way to solve the problem of slow convergence of the series is to transform it into another series which in turn converges faster than the original series. The transformation then becomes the key to success or failure in achieving the accelerated convergence. Acceleration techniques employed so far make use of Kummer's transformation in conjunction with Poisson summation formula [1], [2], and Shank's transform [3]. In this paper we report the use of θ-algorithm [4], [5] in accelerating the convergence of free-space periodic Green's functions involving a single infinite summation. The algorithm is simple to implement and is shown to perform better than Shank's transform [6].

Manuscript received May 13, 1991; revised September 12, 1991.

The authors are with the Department of Electrical Engineering, The University of Tulsa, Tulsa, OK 74104.

IEEE Log Number 9104480.

II. θ-Algorithm

Let S_n be the partial sum of n terms of a series such that $S_n \rightarrow S$ as $n \rightarrow \infty$, where S is the sum of the series. The θ-algorithm can be computed as follows with the even order terms given by

$$\theta_{2k+2}^{(n)} = \theta_{2k}^{(n+1)} + \frac{[\theta_{2k}^{(n+2)} - \theta_{2k}^{(n+1)}]\,[\theta_{2k+1}^{(n+2)} - \theta_{2k+1}^{(n+1)}]}{[\theta_{2k+1}^{(n+2)} - 2\theta_{2k+1}^{(n+1)} + \theta_{2k+1}^{(n)}]}, \quad k = 0, 1, 2, \cdots \tag{1}$$

and the odd order terms by

$$\theta_{2k+1}^{(n)} = \theta_{2k-1}^{(n+1)} + \frac{1}{[\theta_{2k}^{(n+1)} - \theta_{2k}^{(n)}]}, \qquad k = 0, 1, 2, \cdots \tag{2}$$

where

$$\theta_{-1}^{(n)} = 0, \quad \theta_0^{(n)} = S_n. \tag{3}$$

The even order terms, $\theta_{2k+2}^{(n)}$, give estimates of S whereas the odd order terms, $\theta_{2k+1}^{(n)}$, are merely intermediate quantities. The algorithm can be illustrated by applying it to the slowly converging series for $ln2$:

$$ln2 = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}.$$

As indicated, the series converges to $ln2 = 0.6931471$. The results of applying the θ-algorithm to the sequence of partial sums S_1, S_2, $\cdots$, S_{10} are given in Table I. The algorithm converges to seven significant digits (one digit less than the number of digits carried in the partial sum).

III. Free-Space Periodic Green's Functions

The Green's function for a one-dimensional array of phase shifted line sources located at (x', y') in each unit cell and spaced d units apart along the y axis is given by

$$G = \frac{1}{4j} \sum_{m=-\infty}^{+\infty} e^{-jk_y md} H_0^{(2)}(k[(x - x')^2 + (y - y' - md)^2]^{1/2}) \tag{4}$$

where $H_0^{(2)}(\cdot)$ is the zeroth-order Hankel function of the second kind, k is the wavenumber of the medium, k_y is the inter-element phase shift, and the coordinates (x, y) locate the observation point. The spatial domain Green's function in (4) converges very slowly for all combinations of source and observation points. The Fourier transform of (4) gives the spectral domain Green's function given by

$$G = \sum_{m=-\infty}^{\infty} \frac{1}{j2dk_{x_m}} \exp(-jk_{x_m}|x - x'|) \cdot \exp[-j(k_y + 2m\pi/d)(y - y')], \tag{5}$$

TABLE I
APPLICATION OF θ-ALGORITHM TO THE SERIES FOR $ln2$

n	$\theta_0^{(n)} = S_n$	$\theta_2^{(n)}$	$\theta_4^{(n)}$	$\theta_6^{(n)}$
1	1.0000000	0.6944444	0.6931490	0.6931470
2	0.5000000	0.6927083	0.6931465	
3	0.8333333	0.6933332	0.6931473	
4	0.5833333	0.6930555	0.6931470	
5	0.7833333	0.6931972		
6	0.6166666	0.6931175		
7	0.7595238	0.6931657		
8	0.6345238			
9	0.7456349			
10	0.6456349			

where

$$k_{x_m} = \begin{cases} \sqrt{k^2 - (k_y + 2m\pi/d)^2}, & k^2 > (k_y + 2m\pi/d)^2 \\ -j\sqrt{(k_y + 2m\pi/d)^2 - k^2}, & k^2 < (k_y + 2m\pi/d)^2. \end{cases} \quad (6)$$

The spectral domain Green's function converges rapidly for $x \neq x'$. This is due to the exponential factor which aids in the convergence. However for $x = x'$, the series converges very slowly.

The Green's function for a one-dimensional periodic array of point sources located d units apart in the z direction is given by

$$G = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \frac{\exp\{-jk[(x - x')^2 + (y - y)^2 + (z - md)^2]^{1/2}}{[(x - x')^2 + (y - y')^2 + (z - md)^2]^{1/2}}. \quad (7)$$

IV. NUMERICAL RESULTS

The convergence properties of the periodic Green's function series given in (4), (5), and (7) are reported here for various combinations of source and observation points. The results obtained by θ-algorithm are compared with those obtained by the application of Shanks' transform and a direct summation of the series. Since the direct summation of the Green's function series converges very slowly, the results of the direct sum that could fit within the scale chosen, are shown. A convergence criterion defined in [1] is employed here to terminate the summation process. A relative error measure is computed by comparing the result of the algorithm to that of summing the series to machine precision. Without loss of generality, we take $k_y = 0$ and the reference source at the origin, defining $(x', y') = (0, 0)$.

The logarithm of the relative error magnitude versus number of terms for the spatial domain Green's function in (4) is shown in Figs. 1 and 2. The convergence factor, ϵ_c, is indicated alongside each point in the figures. It is shown in Fig. 1 that for $(x, y) = (0.01\lambda, 0.3\lambda)$, the θ-algorithm gives zero relative error in 21 terms for $\epsilon_c = 1 \times 10^{-5}$. This indicates that the algorithm has converged to machine precision. As shown in Fig. 2 for $\epsilon_c = 1 \times 10^{-5}$ the θ-algorithm converges to machine precision in less than 20 terms. The Shanks' transform converges in 53 terms with the relative error approaching 1×10^{-3}. The direct sum converges extremely slowly taking over 100 000 terms to arrive at three significant digit accuracy. The computation time (on VAX 6350) as a function of $1/\epsilon_c$ for the spatial Green's function is shown in Fig. 3. As illustrated, for $(x, y) = (0.01\lambda, 0)$, the θ-algorithm converges in 0.06 s, Shanks' transform in 0.07 s and the direct sum in 40 s for $\epsilon_c = 1 \times 10^{-4}$. This results in a saving in computation time of the order of 600 in using the θ-algorithm over a direct summation of the series.

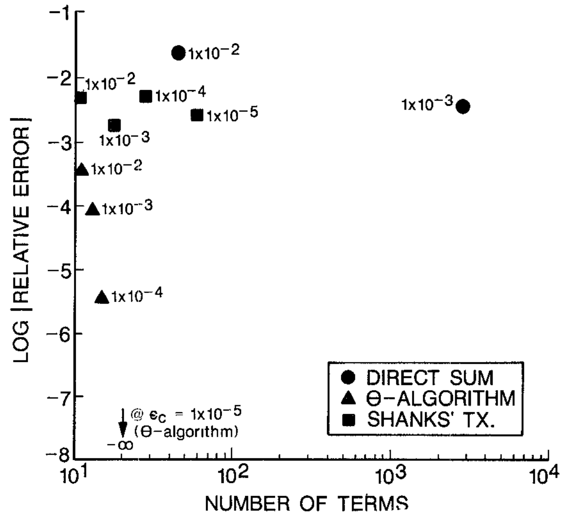

Fig. 1. Log of relative error magnitude versus number of terms for spatial domain Green's function in (4) for $\lambda = 1$ m, $d = 0.6\lambda$, $(x, y) = (0.01\lambda, 0.3\lambda)$.

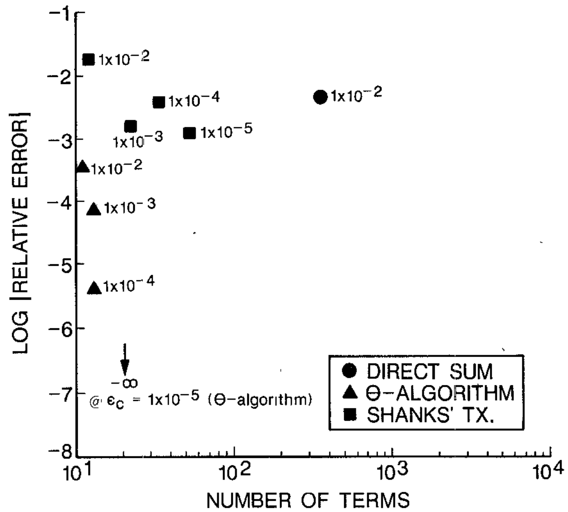

Fig. 2. Log of relative error magnitude versus number of terms for spatial domain Green's function in (4) for $\lambda = 1$ m, $d = 0.6\lambda$ $(x, y) = (0.1\lambda, 0.3\lambda)$.

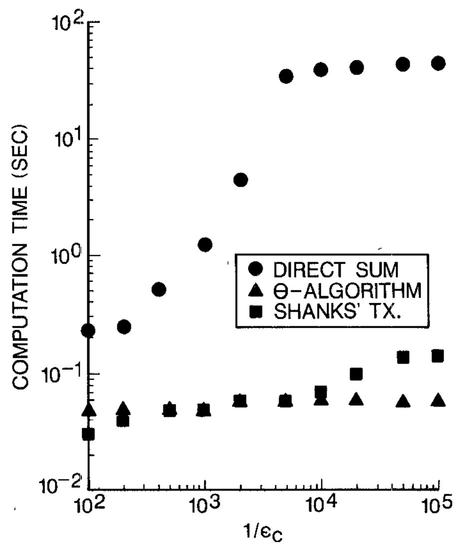

Fig. 3. Computation time versus $1/\epsilon_c$ for spatial domain Green's function in (5) for $\lambda = 1$ m, $d = 0.6\lambda$, $(x, y) = (0.01\lambda, 0.0\lambda)$.

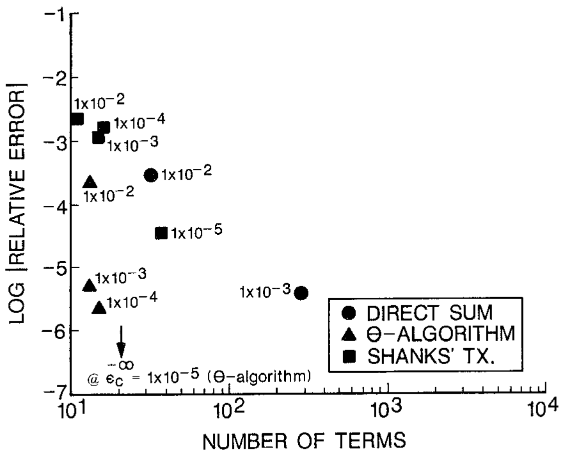

Fig. 4. Log of relative error magnitude versus number of terms of spectral domain Green's function in (5) for $\lambda = 1$ m, $d = 1.2\lambda$;, $(x, y) = (0, 0.6\lambda)$.

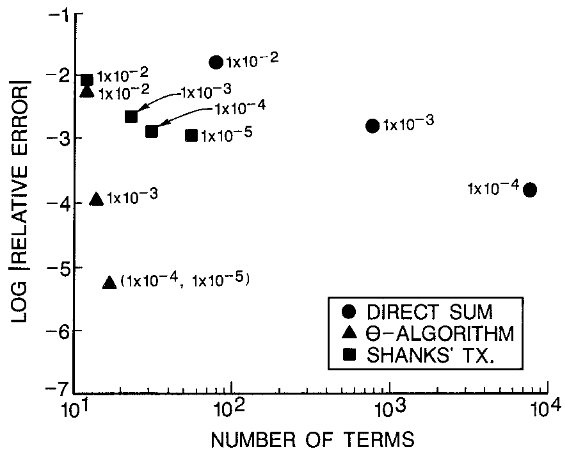

Fig. 5. Log of relative error magnitude versus number of terms for spectral domain Green's function in (5) for $\lambda = 1$ m, $d = 1.2\lambda$, $(x, y) = (0, 0.3\lambda)$.

The logarithm of the relative error versus number of terms for the spectral domain Green's function given in (5) is shown in Figs. 4 and 5. As shown in Fig. 4, the θ-algorithm converges to machine precision in 20 terms for $\epsilon_c = 1 \times 10^{-5}$. It is shown in Fig. 5 that the direct sum converges very slowly taking several thousand terms to achieve four significant digit accuracy. On the other hand, the θ-algorithm converges to five significant digits in 17 terms for $\epsilon_c = 1 \times 10^{-4}$.

The magnitude of the relative error versus the number of terms for the periodic Green's function in (7) is shown in Figs. 6 and 7. The spectacular convergence rate of the θ-algorithm is illustrated in Fig. 6 for $(x, y, z) = (0.2\lambda, 0.1\lambda, 0.3\lambda)$. In this case for $\epsilon_c = 1 \times 10^{-5}$ the direct sum takes 12 000 terms, Shanks' transform takes 43 terms and the θ-algorithm converges in merely 21 terms. A similar result is shown in Fig. 7 for $(x, y, z) = (0.1\lambda, 0.1\lambda, 0.3\lambda)$. For $\epsilon_c = 1 \times 10^{-5}$, the direct sum converges in 23 000 terms, the Shanks' transform in 40 terms and the θ-algorithm in just 19 terms. Besides converging in fewer number of terms the θ-algorithm has the least error. As the observation point is taken closer to the reference source point at the origin, the direct sum converges extremely slowly. Even in such cases the θ-algorithm has the fastest convergence. This is illustrated in Fig. 8 in which the computation time (on VAX 6350) versus $1/\epsilon_c$ is plotted for $(x,$

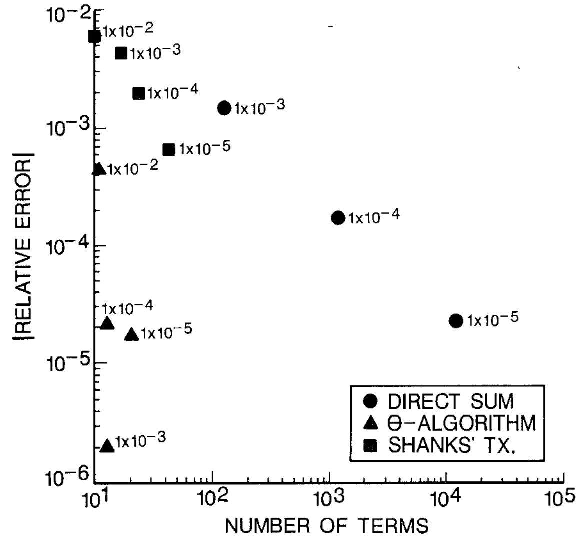

Fig. 6. Relative error magnitude versus number of terms for the Green's function series in (7) for $\lambda = 1$ m, $d = 0.6\lambda$, $(x, y, z) = (0.2\lambda, 0.1\lambda, 0.3\lambda)$.

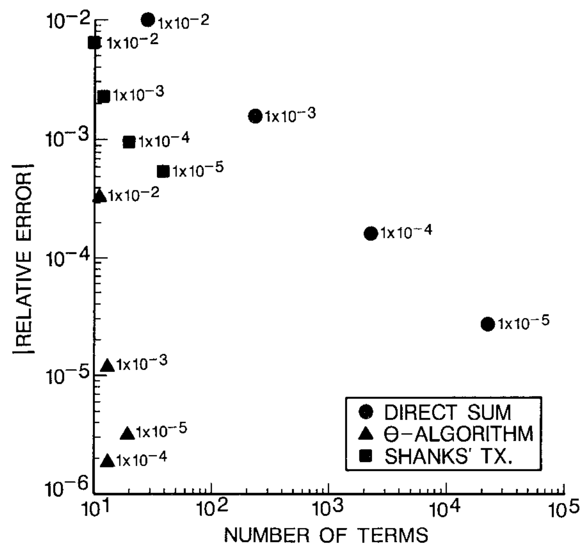

Fig 7. Relative error magnitude versus number of terms for the Green's function series in (7) for $\lambda = 1$ m, $d = 0.6\lambda$, $(x, y, z) = (0.1\lambda, 0.1\lambda, 0.3\lambda)$.

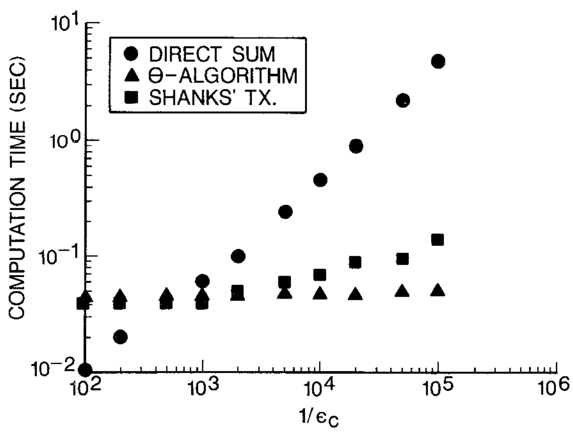

Fig. 8. Computation time versus $1/\epsilon_c$ for the Green's function series in (7) for $\lambda = 1$ m, $d = 0.6\lambda$, $(x, y, z) = (0, 0, 0.3\lambda)$.

$y, z) = (0, 0, 0.3\lambda)$. The saving in computation time in using the θ-algorithm over a direct summation of the series is of the order of 100.

V. Conclusion

The use of θ-algorithm is shown to have a dramatic impact in accelerating the convergence of slowly converging series. The algorithm has been applied with success to the series representing the free-space periodic Green's functions. Numerical results indicate that the algorithm is superior to Shanks' transform both in convergence and speed. In most cases the algorithm converges to a high degree of precision in about 20 terms. This is indeed remarkable as a direct sum of the series converges extremely slowly. The use of θ-algorithm results in a considerable amount of saving in computation time thereby increasing the computational efficiency in problem involving one-dimensional periodicity.

References

[1] S. Singh, W. F. Richards, J. R. Zinecker, and D. R. Wilton, "Accelerating the convergence of series representing the free space periodic Green's function," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 1958-1962, Dec. 1990.

[2] R. E. Jorgenson and R. Mittra, "Efficient calculation of the free-space periodic Green's function," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 633-642, May 1990.

[3] S. Singh and R. Singh, "Application of transforms to accelerate the summation of periodic free-space Green's functions," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 1746-1748, Nov. 1990.

[4] C. Brezinski, "Acceleration de suites a convergence logarithmique," *C.R. Acad. Sci. Paris Ser. A-B*, vol. 273, pp. A727-A730, 1971.

[5] C. Brezinski, "Some new convergence acceleration methods," *Mathematics of Computation*, vol. 39, no. 159, pp. 133-145, July 1982.

[6] D. Shanks, "Non-linear transformations of divergent and slowly convergent sequences," *J. Math. Physics*, vol. 34, pp. 1-42, 1955.

On the Use of Chebyschev–Toeplitz Algorithm in Accelerating the Numerical Convergence of Infinite Series

Surendra Singh and Ritu Singh

***Abstract*—It is shown here that a simple application of the Chebyschev–Toeplitz algorithm enhances the rate of convergence of slowly converging series. The algorithm is applied to series representing the periodic Green's functions involving a single infinite summation. The algorithm yields highly accurate results within relatively fewer terms. A quantitative comparison is shown with methods previously reported in the literature.**

I. Introduction

The computation of electromagnetic radiation or scattering from a periodic geometry involves the summation of a Green's function series which converges very slowly. The summation of the series may be accelerated by transforming the series such that the new series converges rapidly [1]–[5]. The transformation, however, requires analytical work which is characteristic for each series. This in some sense limits the applicability of the method. It is our intent to demonstrate that algorithms [6]–[11] which can be readily applied to any slowly converging series, irrespective of its functional form, are highly accurate and efficient. In particular, we report the use of the Chebyschev–Toeplitz (CT) algorithm [11] in accelerating the convergence of periodic Green's function series.

Manuscript received May 16, 1991; revised August 13, 1991.

The authors are with the Department of Electrical Engineering, The University of Tulsa, Tulsa, OK 74104.

IEEE Log Number 9103905.

II. Chebyschev–Toeplitz (CT) Algorithm

Let S_n be the partial sum of n terms of a series such that $S_n \to S$ as $n \to \infty$, where S is the sum of the series. The CT algorithm is defined by the following equations [11]:

$$t_{-1}^{(n)} = 0, \quad t_0^{(n)} = S_n, \quad \sigma_0 = 1, \tag{1}$$

$$t_1^{(n)} = t_0^{(n)} + 2t_0^{(n+1)}, \quad \sigma_1 = 3, \tag{2}$$

$$t_{k+1}^{(n)} = 2t_k^{(n)} + 4t_k^{(n+1)} - t_{k-1}^{(n)}, \qquad k = 1, 2, \cdots \tag{3}$$

$$\sigma_{k+1} = 6\sigma_k - \sigma_{k-1}, \qquad k = 1, 2, \cdots \tag{4}$$

$$T_k^{(n)} = t_k^{(n)}/\sigma_k, \qquad k = 0, 1, 2, \cdots. \tag{5}$$

The nth iterate of the CT algorithm is given by $T_k^{(n)}$, which gives an estimate of the sum of the series. The algorithm can be illustrated by applying it to the slowly converging Liebniz series for π:

$$\pi = \sum_{m=0}^{\infty} \frac{4(-1)^m}{2m+1}. \tag{6}$$

The result of applying the CT algorithm to the sequence of partial sums $S_0, S_1, \cdots, S_8$ is given in Table I. The algorithm converges to six significant digits. Although the even and odd order iterates of the CT algorithm provide an estimate of S, only the even orders are shown in the table.

III. Free-Space Periodic Green's Functions

The spectral domain Green's function for a one-dimensional array of line source spaced d units apart in the x direction is given by

$$G = \sum_{m=-\infty}^{\infty} \frac{1}{j2dk_{y_m}} \cdot \exp(-jk_{y_m}|y-y'| \exp[-j(2m\pi/d)(x-x')] \tag{7}$$

where

$$k_{y_m} = \begin{cases} \sqrt{k^2 - (2m\pi/d)^2}, & k^2 > (2m\pi/d)^2 \\ -j\sqrt{(2m\pi/d)^2 - k^2}, & k^2 < (2m\pi/d)^2 \end{cases}$$

k is the wave number of the medium, (x', y') locates the reference source and (x, y) locates the observation point. The series in (7) converges very slowly whenever $y = y'$. This is referred to as the "on plane" case. The spatial domain counterpart of the periodic Green's function in (7) is given by

$$G = \sum_{m=-\infty}^{\infty} H_0^{(2)}(k[(y-y')^2 + (x-x'-md)^2]^{1/2}) \tag{8}$$

where $H_0^{(2)}$ is the zeroth-order Hankel function of the second kind.

The Green's function for a one-dimensional array of point sources